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TECHNICAL NOTE

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A METHOD OF CALCULATING A STABILITY BOUNDARY THAT DEFINES A REGION OF SATISFACTORY PERIOD-DAMPING RELATIONSHIP OF THE OSCILLATORY MODE OF MOTION

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Vertical Wind Tunnel

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SUMMARY

A method has been derived by which a boundary can be obtained that defines a region in which there exists a satisfactory relationship between the period and damping of the lateral oscillatory mode of motion, according to any given criterion for this relationship. In addition, a method is discussed by which curves representing a constant rate of spiral divergence may be constructed.

The methods as presented are applicable to both lateral-stability and longitudinal-stability analyses.

INTRODUCTION

In lateral-stability analyses it is the usual practice to calculate a neutral-oscillatory-stability boundary that is plotted as a function of the directional-stability derivative C_{n_β} and the effective-dihedral derivative C_{l_β}. This boundary has a special significance, since for a particular airplane it indicates the combinations of C_{n_β} and C_{l_β} necessary for oscillatory stability. Although this boundary is definitely an aid in a stability analysis, its value is somewhat restricted in that it affords no information about the variation of the period-damping relationship throughout the stable region. That is, that region of the C_{n_β},C_{l_β} plane in which a given criterion for satisfactory relationship of the period and damping of the oscillatory mode will be satisfied cannot be determined from this boundary. This information can be obtained by using the methods described in reference 1 to calculate curves of constant period and constant damping, but because of the complexity of the calculations involved, a more direct approach to the problem is desirable. In the present paper a method is derived by which it is possible to obtain directly a boundary, plotted as a function of C_{n_β} and C_{l_β}, that will define a region in which there exists a satisfactory relationship between the period and damping of the oscillatory mode of motion, according to a prescribed criterion for this relationship.

For the combinations of C_{n_β} and C_{l_β} necessary to satisfy various criterions for the period and damping of the oscillatory mode, undesirable spiral instability may be present. In view of this fact a method is also presented by which curves representing a constant rate of spiral divergence may be constructed.

SYMBOLS AND COEFFICIENTS

ϕ angle of roll, radians

ψ angle of yaw, radians

β angle of sideslip, radians (v/V)

v sideslip velocity along the Y-axis, feet per second

V airspeed, feet per second

ρ mass density of air, slugs per cubic foot

q dynamic pressure, pounds per square foot $\left(\frac{1}{2}\rho V^2\right)$

b wing span, feet

S wing area, square feet

W weight of airplane, pounds

m mass of airplane, slugs (W/g)

g acceleration due to gravity, feet per second per second

μ_b relative-density factor $(m/\rho Sb)$

η inclination of principal longitudinal axis of airplane with respect to flight path, positive when principal axis is above flight path at the nose, degrees

γ angle of flight path to horizontal axis, positive in a climb, degrees

k_{X_0} radius of gyration in roll about principal longitudinal axis, feet

k_{Z_0} radius of gyration in yaw about principal vertical axis, feet

K_{X_0} nondimensional radius of gyration in roll about principal longitudinal axis $\left(k_{X_0}/b\right)$

K_{Z_0} nondimensional radius of gyration in yaw about principal vertical axis $\left(k_{Z_0}/b\right)$

K_X nondimensional radius of gyration in roll about longitudinal stability axis $\left(\sqrt{K_{X_0}{}^2\cos^2\eta + K_{Z_0}{}^2\sin^2\eta}\right)$

K_Z nondimensional radius of gyration in yaw about vertical stability axis $\left(\sqrt{K_{Z_0}{}^2\cos^2\eta + K_{X_0}{}^2\sin^2\eta}\right)$

K_{XZ} nondimensional product-of-inertia parameter $\left(\left(K_{Z_0}{}^2 - K_{X_0}{}^2\right)\sin\eta\cos\eta\right)$

C_L trim lift coefficient $\left(\frac{W\cos\gamma}{qS}\right)$

C_l rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{qSb}\right)$

C_n yawing-moment coefficient $\left(\frac{\text{Yawing moment}}{qSb}\right)$

C_Y lateral-force coefficient $\left(\frac{\text{Lateral force}}{qS}\right)$

C_{l_β} effective-dihedral derivative, rate of change of rolling-moment coefficient with angle of sideslip, per radian $(\partial C_l/\partial\beta)$

C_{n_β} directional-stability derivative, rate of change of yawing-moment coefficient with angle of sideslip, per radian $(\partial C_n/\partial\beta)$

C_{Y_β} lateral-force derivative, rate of change of lateral-force coefficient with angle of sideslip, per radian $(\partial C_Y/\partial\beta)$

C_{n_r} damping-in-yaw derivative, rate of change of yawing-moment coefficient with yawing-angular-velocity factor, per radian $\left(\partial C_n/\partial\frac{rb}{2V}\right)$

C_{n_p} rate of change of yawing-moment coefficient with rolling-angular-velocity factor, per radian $\left(\partial C_n/\partial\frac{pb}{2V}\right)$

C_{l_p} damping-in-roll derivative, rate of change of rolling-moment coefficient with rolling-angular-velocity factor, per radian $\left(\partial C_l/\partial\frac{pb}{2V}\right)$

C_{l_r} rate of change of rolling-moment coefficient with yawing-angular-velocity factor, per radian $\left(\partial C_l / \partial \frac{rb}{2V}\right)$

C_{Y_p} rate of change of lateral-force coefficient with rolling-angular-velocity factor, per radian $\left(\partial C_Y / \partial \frac{pb}{2V}\right)$

C_{Y_r} rate of change of lateral-force coefficient with yawing-angular-velocity factor, per radian $\left(\partial C_Y / \partial \frac{rb}{2V}\right)$

C_{l_ϕ} rate of change of rolling-moment coefficient with angle of roll, per radian $(\partial C_l / \partial \phi)$

t time, seconds

s_b nondimensional time parameter based on span (Vt/b)

D_b differential operator $\left(\frac{d}{ds_b}\right)$

R_1 Routh's discriminant

λ_1 complex root of stability equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \qquad \left(\lambda_1 = a + i\omega = Re^{i\theta}\right)$$

$R = \sqrt{a^2 + \omega^2}$

$\theta = \tan^{-1} \frac{\omega}{a}$

ϵ_n, ϕ_n factors used in calculation of boundary of satisfactory period-damping relationship $\left(\epsilon_n = \frac{\cos n\theta}{R^n}, \phi_n = \frac{\sin n\theta}{R^{n-1}} \text{ for } 0 \leqq n \leqq 4\right)$

λ_2 spiral stability root of stability equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

P period of oscillation, seconds

$T_{1/2}$ time for amplitude of oscillatory or spiral mode to decrease to one-half its original value

T_2 time for amplitude of oscillatory or spiral mode to increase to double its original value

A,B,C,D,E coefficients of lateral-stability equation

EQUATIONS OF MOTION

The nondimensional linearized equations of motion, referred to the stability axes, used to calculate the spiral-stability and oscillatory-stability boundaries for any flight condition, are:

Rolling

$$2\mu_b\left(K_X{}^2D_b{}^2\phi + K_{XZ}D_b{}^2\psi\right) = C_{l_\beta}\beta + \frac{1}{2}C_{l_r}D_b\psi + \frac{1}{2}C_{l_p}D_b\phi$$

Yawing

$$2\mu_b\left(K_Z{}^2D_b{}^2\psi + K_{XZ}D_b{}^2\phi\right) = C_{n_\beta}\beta + \frac{1}{2}C_{n_r}D_b\psi + \frac{1}{2}C_{n_p}D_b\phi$$

Sideslipping

$$2\mu_b\left(D_b\beta + D_b\psi\right) = C_{Y_\beta}\beta + \frac{1}{2}C_{Y_p}D_b\phi + C_L\phi + \frac{1}{2}C_{Y_r}D_b\psi + \left(C_L \tan\gamma\right)\psi$$

When $\phi_o e^{\lambda s_b}$ is substituted for ϕ, $\psi_o e^{\lambda s_b}$ for ψ, and $\beta_o e^{\lambda s_b}$ for β in the equations written in determinant form, λ must be a root of the stability equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \tag{1}$$

where

$$A = 8\mu_b{}^3\left(K_X{}^2K_Z{}^2 - K_{XZ}{}^2\right)$$

$$B = -2\mu_b{}^2\left(2K_X{}^2K_Z{}^2C_{Y_\beta} + K_X{}^2C_{n_r} + K_Z{}^2C_{l_p} - 2K_{XZ}{}^2C_{Y_\beta} - K_{XZ}C_{l_r} - K_{XZ}C_{n_p}\right)$$

$$C = \mu_b\left(K_X{}^2C_{n_r}C_{Y_\beta} + 4\mu_bK_X{}^2C_{n_\beta} + K_Z{}^2C_{l_p}C_{Y_\beta} + \frac{1}{2}C_{n_r}C_{l_p} - K_{XZ}C_{l_r}C_{Y_\beta}\right.$$

$$- 4\mu_bK_{XZ}C_{l_\beta} - C_{n_p}K_{XZ}C_{Y_\beta} - \frac{1}{2}C_{n_p}C_{l_r} + K_{XZ}C_{n_\beta}C_{Y_p} - K_Z{}^2C_{Y_p}C_{l_\beta}$$

$$\left.- K_X{}^2C_{Y_r}C_{n_\beta} + K_{XZ}C_{Y_r}C_{l_\beta}\right)$$

$$D = -\frac{1}{4}C_{n_r}C_{l_p}C_{Y_\beta} - \mu_b C_{l_p}C_{n_\beta} + \frac{1}{4}C_{n_p}C_{l_r}C_{Y_\beta} + \mu_b C_{n_p}C_{l_\beta} + 2\mu_b C_L K_{XZ} C_{n_\beta}$$

$$- 2\mu_b C_L K_Z{}^2 C_{l_\beta} - 2\mu_b K_X{}^2 C_{n_\beta} C_L \tan\gamma + 2\mu_b K_{XZ} C_{l_\beta} C_L \tan\gamma$$

$$+ \frac{1}{4}C_{l_p}C_{n_\beta}C_{Y_r} - \frac{1}{4}C_{n_p}C_{l_\beta}C_{Y_r} - \frac{1}{4}C_{l_r}C_{n_\beta}C_{Y_p} + \frac{1}{4}C_{n_r}C_{l_\beta}C_{Y_p}$$

$$E = \frac{1}{2}C_L\left(C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}\right) + \frac{1}{2}C_L \tan\gamma\left(C_{l_p}C_{n_\beta} - C_{n_p}C_{l_\beta}\right)$$

The damping and period of the lateral oscillation in seconds are given by the equations

$$T_{1/2} = \frac{-0.693}{a}\frac{b}{V}$$

$$T_2 = \frac{0.693}{a}\frac{b}{V}$$

$$P = \frac{6.28}{\omega}\frac{b}{V}$$

where a and ω are the real and imaginary parts of the complex root of stability equation (1). The damping of the spiral mode of motion in seconds is given by the equations

$$T_{1/2} = \frac{-0.693}{\lambda_2}\frac{b}{V}$$

$$T_2 = \frac{0.693}{\lambda_2}\frac{b}{V}$$

ANALYSIS

Method for constructing a boundary that defines a region of satisfactory period-damping relationship.- For the derivation of a method by which the period-damping relationship of the lateral oscillatory mode may be expressed as a function of C_{n_β} and C_{l_β}, the following assumptions are made:

1. The stability equation is given in the conventional form $A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$

2. The coefficients A, B, C, D, and E are functions of the stability derivatives, all of which have prescribed values with the exception of C_{n_β} and C_{l_β}.

3. The stability equation has for one of its roots

$$\lambda_1 = a + i\omega = Re^{i\theta}$$

where

$$\theta = \tan^{-1}\frac{\omega}{a}$$

and

$$R = \sqrt{a^2 + \omega^2}$$

This root λ_1 represents an oscillatory mode of motion, the period of which is given by $P = \frac{2\pi}{\omega}\frac{b}{V}$ and the time required for the amplitude of the oscillation to damp to one-half its original value is given by $T_{1/2} = \frac{-0.693}{a}\frac{b}{V}$. Substituting $Re^{i\theta}$ for λ in the stability equation, and noting that $e^{in\theta} = \cos n\theta + i \sin n\theta$, results in the expression

$$AR^4(\cos 4\theta + i \sin 4\theta) + BR^3(\cos 3\theta + i \sin 3\theta) + CR^2(\cos 2\theta + i \sin 2\theta) + DR(\cos \theta + i \sin \theta) + E = 0 \tag{2}$$

Equating to zero real and imaginary parts in equation (2) gives the equations

$$\left.\begin{aligned} AR^4\cos 4\theta + BR^3\cos 3\theta + CR^2\cos 2\theta + DR \cos \theta + E = 0 \\ AR^4\sin 4\theta + BR^3\sin 3\theta + CR^2\sin 2\theta + DR \sin \theta = 0 \end{aligned}\right\} \tag{3}$$

Multiplying the first equation of equations (3) by $\cos 4\theta$, the second by $\sin 4\theta$, and adding the two resulting equations give the expression

$$A\epsilon_0 + B\epsilon_1 + C\epsilon_2 + D\epsilon_3 + E\epsilon_4 = 0 \tag{4}$$

where

$$\epsilon_n = \frac{\cos n\theta}{R^n}$$

Similarly, multiplying the first equation of equations (3) by $\sin 4\theta$, the second by $-\cos 4\theta$, and adding the two resulting equations give the expression

$$A\phi_0 + B\phi_1 + C\phi_2 + D\phi_3 + E\phi_4 = 0 \tag{5}$$

where

$$\phi_n = \frac{\sin n\theta}{R^{n-1}}$$

Since θ and R are functions of the terms a and ω of the root λ_1, the values of the factors ϵ_n of equation (4) and ϕ_n of equation (5) will be fixed for specified conditions of period and damping. Therefore, solving equations (4) and (5) simultaneously results in the C_{n_β}, C_{l_β} combination that satisfies any desired period and damping relationship of the oscillatory mode.

If a suitable criterion exists for the damping of the lateral oscillation expressed as a function of the period, for example,

$$T_{1/2} = f(P)$$

this criterion will be expressible in terms of a and ω by an equation of the form

$$\frac{-0.693}{a}\frac{b}{V} = f\left(\frac{2\pi}{\omega}\frac{b}{V}\right)$$

Thus, for any given period, there exists a corresponding value of the damping term a necessary to satisfy the criterion. The terms ϵ_n of equation (4) and terms ϕ_n of equation (5) are expressed in terms of R and θ, which are functions of a and ω $\left(R = \sqrt{a^2 + \omega^2}\right.$ and $\left.\theta = \tan^{-1}\frac{\omega}{a}\right)$; therefore, curves showing the variation of ϵ_n and ϕ_n with the period of the oscillation that will satisfy the given criterion can be constructed. For any value of the period P the values of ϵ_n and ϕ_n may be taken from these curves and substituted in equations (4) and (5). The corresponding C_{n_β}, C_{l_β} values may then be determined from a simultaneous solution of these equations.

It is important to note that the method described is applicable to any criterion for satisfactory damping of the oscillatory mode expressed as a function of the period. If the criterion is expressed in terms of the number of cycles required to damp to half amplitude, that is, $C_{1/2} = f(P)$, the preceding method can still be applied since $C_{1/2} = \frac{T_{1/2}}{P}$.

General procedure for use of the method.- In the analysis of a given airplane the coefficients A, B, C, D, and E of equation (1) are evaluated in terms of the known or prescribed values of all the stability derivatives except two $\left(\text{usually } C_{n_\beta} \text{ and } C_{l_\beta}\right)$ which are left in symbolic form. A point on a curve, plotted as a function of the two derivatives chosen to be variables, defining the region of satisfactory period-damping relationship is obtained in the following manner: For an arbitrary value of the period P, values of ϵ_n and ϕ_n are taken from previously constructed curves that satisfy the desired criterion for period-damping relationship of the oscillation. These values are substituted in equations (4) and (5), together with the expressions for the coefficients A, B, C, D, and E. The two equations are then solved simultaneously for the variable derivatives. By repeating the process for successive values of P the desired boundary will be obtained. In order that the significance of the boundary thus obtained be clearly understood it is necessary to know on which side of this boundary there exists a satisfactory period-damping relationship according to the prescribed criterion. A method for obtaining this information is discussed in appendix A.

Application of the method presented to longitudinal-stability analyses.- It should be pointed out that the methods presented herein are not limited to an analysis of lateral stability but can be readily employed in the longitudinal-stability analysis as well. If a suitable criterion exists for the period-damping relationship of the longitudinal oscillatory mode of motion, curves of ϕ_n and ϵ_n plotted against P can be constructed and points on a boundary defining the region in which this satisfactory relationship exists between the period and damping of the longitudinal oscillation can be calculated. The coefficients A, B, C, D, and E to be used in the simultaneous solution of equations (4) and (5) for the longitudinal case must, of course, be the coefficients of the longitudinal-stability biquadratic. The variables of interest become, instead of C_{n_β} and C_{l_β}, any two appropriate parameters for longitudinal stability.

Method for constructing curves of constant rate of spiral divergence.- For the combinations of C_{n_β} and C_{l_β} necessary to satisfy various criterions for the period and damping of the oscillatory mode, undesirable spiral instability may be present. In view of this fact it is desirable to know something about the divergence gradient throughout the C_{n_β},C_{l_β} plane.

The usual method of obtaining this information involves the solution of a series of stability biquadratics, often a rather laborious procedure. The following method is presented which gives the desired information with a minimum of calculations.

Assume that the lateral-stability biquadratic (equation (1)) has as one of its roots $\lambda = \lambda_2$. This substitution for λ in equation (1) results in the expression

$$A\lambda_2^4 + B\lambda_2^3 + C\lambda_2^2 + D\lambda_2 + E = 0 \tag{6}$$

where the coefficients A, B, C, D, and E are functions of C_{n_β} and C_{l_β} as before. If $\lambda_2 = 0$, equation (6) is reduced to the equation $E = 0$, which defines the boundary of neutral spiral stability. However, if λ_2 is assigned a finite value representing a given rate of divergence, equation (6) becomes a definite function of C_{n_β} and C_{l_β}. The solution of this resulting equation for a sequence of values of C_{n_β} gives the corresponding values of C_{l_β} on a curve representing a constant rate of divergence of the spiral mode of motion. These curves can be constructed for as many values of λ_2 as may be necessary for an adequate analysis of the spiral instability of a particular airplane.

ILLUSTRATIVE EXAMPLES AND DISCUSSION

In order to meet the present Navy-Air Force flying-qualities requirement (references 2 and 3) the period and damping of the lateral oscillation must satisfy the following criterion:

$$\left.\begin{aligned} T_{1/2} &= 1.5 && (0 \leqq P \leqq 2) \\ T_{1/2} &= 2.5P - 3.5 && (2 \leqq P \leqq \infty) \end{aligned}\right\} \tag{7}$$

This criterion is illustrated in figure 1.

As an example of the methods presented, calculations were made for the hypothetical high-speed airplane described in table I by using equation (7) as the criterion for satisfactory damping of the lateral oscillation. The values of the factors $\dfrac{\epsilon_n}{(V/b)^n}$ and $\dfrac{\phi_n}{(V/b)^{n-1}}$ for a sequence of values of the period P are presented in table II and the variations of these factors with P are shown in figure 2. Note that the ordinates in figure 2 are $\dfrac{\epsilon_n}{(V/b)^n}$ and $\dfrac{\phi_n}{(V/b)^{n-1}}$ rather than ϵ_n and ϕ_n in order that the curves in figure 2 will apply for any airplane at any flight condition if the criterion described in equation (7) is used. For a number of values of P, the values of ϵ_n and ϕ_n were determined from these curves and were then substituted in equations (4) and (5), and the corresponding C_{n_β}, C_{l_β} values were obtained from a simultaneous solution for these equations.

The boundary, plotted as a function of C_{n_β} and C_{l_β}, defining the region that satisfies the period-damping relationship given by equation (7) is presented in figure 3. For purpose of comparison the neutral-oscillatory-stability boundary ($R_1 = 0$) for this airplane is also plotted in figure 3.

From inspection of the $R_1 = 0$ boundary this airplane is seen to be oscillatorily stable for almost any combination of C_{n_β} and C_{l_β} shown in the figure. This boundary, however, gives no indication of the degree of stability present at any point in the quadrant. The curve defining the Navy-Air Force criterion for satisfactory damping of the lateral oscillation clearly shows that the criterion is satisfied in only a relatively small part of the stable region of the C_{n_β},C_{l_β} plane. Figure 3, therefore, clearly illustrates that the stability of a given airplane cannot be accurately evaluated by merely constructing the $R_1 = 0$ boundary and noting on which side of this boundary stability exists.

Additional calculations were made in order to obtain curves of constant rate of spiral divergence for this airplane, and the results are presented in figure 4. Should a criterion for the rate of spiral divergence be established, it will then be sufficient in a given spiral-stability analysis to plot only the curve representing that rate of divergence.

The roots of the lateral-stability biquadratic usually indicate the presence of one periodic and two aperiodic modes of motion. If this be true, only one curve defining the region of satisfactory period-damping relationship of the periodic mode would exist. If, however, the roots indicate the presence of two periodic modes of motion, two or more branches of this curve may exist.

Thus, reference 4 shows that for an airplane equipped with an automatic pilot that applies aileron control proportional to the displacement in roll, the roots of the stability equation will represent two oscillatory modes of motion. Calculations were made for this configuration by using the stability derivatives and mass characteristics presented in table I of reference 4. Points on the boundary of satisfactory period-damping relationship were calculated for $C_{l_\beta} = -0.10$ by using as variables the directional-stability derivative C_{n_β} and the derivative C_{l_ϕ} due to the automatic pilot. The equations and coefficients presented in the section entitled "Equations of Motion" were modified to take into account the derivative C_{l_ϕ} in the same manner as was done in reference 4.

The results of these calculations are presented in figure 5. For a number of points in the C_{n_β},C_{l_ϕ} plane, roots of the stability equation were calculated in an effort to understand more clearly the significance of each of the branches of the boundary presented in this figure. The results of these calculations are given in table III. At point A the roots represent two periodic modes, neither satisfying the Navy-Air Force criterion. Upon passing through the branch of the boundary in the upper quadrant to point B, the period-damping relationship of one of the periodic modes becomes satisfactory while the other remains unsatisfactory. At point C the relationship of the unsatisfactory modes has improved somewhat and the other mode remains satisfactory. At point D the previously satisfactory periodic mode has become two aperiodic modes, one of which is unstable. The period-damping relationship of the periodic mode is now satisfactory. At point E the periodic mode has again become unsatisfactory and the two aperiodic modes are stable. Thus, the only part of the C_{n_β},C_{l_ϕ} plane that completely satisfies the Navy-Air Force criterion is that area on the unhatched side of the branch of the boundary appearing in the lower quadrant. It should be pointed out that although the oscillatory mode satisfies the Navy-Air Force criterion within this area, the instability of one of the aperiodic modes is such that the region may be of little practical value.

CONCLUSIONS

The following conclusions were made from a theoretical investigation to develop a method of calculating a stability boundary that defines a region of satisfactory period-damping relationship of the oscillatory mode of motion:

1. Through use of the methods presented a boundary can be obtained that defines a region in which there exists a satisfactory relationship between the period and damping of the lateral oscillatory mode of motion, according to any given criterion for this relationship.

2. A method is also presented by which curves representing a constant rate of spiral divergence may be constructed.

3. These methods, which are developed in detail herein for the analysis of lateral stability, are adaptable as well to the analysis of longitudinal stability.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

A METHOD FOR DETERMINING ON WHICH SIDE OF THE BOUNDARY OF SATISFACTORY PERIOD-DAMPING RELATIONSHIP A GIVEN CRITERION IS SATISFIED

If the stability biquadratic has for one of its roots $\lambda_1 = a + i\omega$, it can be shown from reference 1 that the following parametric equations are satisfied

$$f_1 - \mu f_3 = 0 \tag{A1}$$

$$f - \mu f_2 + \mu^2 f_4 = 0 \tag{A2}$$

where

$$\mu = \omega^2$$

$$f = Aa^4 + Ba^3 + Ca^2 + Da + E$$

$$f_1 = 4Aa^3 + 3Ba^2 + 2Ca + D$$

$$f_2 = 6Aa^2 + 3Ba + C$$

$$f_3 = 4Aa + B$$

$$f_4 = A$$

The coefficients A, B, C, D, and E appearing in the expressions for f, f_1, f_2, f_3, and f_4 are those of the characteristic stability equation. These coefficients are functions of the stability derivatives, two of which are assumed to be variable.

If the parameter μ is eliminated between equations (A1) and (A2), the expression obtained is

$$(f_2 f_3 - f_1 f_4) f_1 - f_3^2 f = 0 \tag{A3}$$

This equation represents a curve of constant damping a, and the frequency of the oscillation at any point on the curve can be calculated by using equation (A1). If $a = 0$, equation (A3) reduces to the familiar expression for the neutral-oscillatory-stability boundary $(BC - AD)D - B^2E = 0$.

Now, for every point on the boundary of satisfactory period and damping, plotted as a function of any two arbitrary derivatives x and y, a root $\lambda_1 = a + i\omega$ will exist that exactly satisfies the prescribed criterion. Assume that at a particular point on the boundary (x_1,y_1) the root is $\lambda_1 = a_1 + i\omega_1$. If $a = a_1$ and $x = x_2 = x_1 + \Delta x$ be substituted in equation (A3), the value $y = y_2$, which is located on a curve of constant damping, can be calculated. By substituting a_1, x_2, and y_2 for a, x, and y, respectively, in equation (A1) the value of μ, and hence ω_2, is determined at the point (x_2,y_2). If the root $a_1 + i\omega_2$ at point (x_2,y_2) satisfies the prescribed criterion, the region in which this point is located is the satisfactory region with respect to the boundary defining the criterion.

An imaginary value of ω_2 at point (x_2,y_2) indicates that no complex root with the real part equal to a_1 exists at this point, since ω_2 must be real if the root $a_1 + i\omega_2$ is to represent an oscillation. That is, $a_1 \pm i\omega_2$ will represent two real roots and before the point (x_2,y_2) can be established as satisfactory or unsatisfactory, the other two roots of the stability equation must be determined. However, if Δx is chosen small enough, ω_2 will be real.

In general, the satisfactory region may also be readily identified if roots of the stability biquadratic are calculated at several points of interest. A method of evaluating the roots of a quartic equation is presented in appendix B.

APPENDIX B

METHODS FOR EVALUATING THE ROOTS OF A QUARTIC EQUATION

Various methods exist for evaluating the roots of high-order polynomials (references 5 and 6). Many of these methods, although highly accurate, become rather laborious in actual application.

A method developed by Lin (reference 5) and independently by Doris Cohen of the NACA (unpublished) affords a means of evaluating roots of high-order equations with a minimum of computation. This method is basically one of synthetic division and under certain conditions, which are discussed subsequently, the desired roots can be obtained very rapidly.

In order to illustrate the method consider the quartic equation

$$\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

As a first approximation, assume that the equation has the quadratic factor $\lambda^2 + j\lambda + k$ where $j = \frac{D}{C}$ and $k = \frac{E}{C}$. Division may then be performed as follows:

$$\begin{array}{r|l}
 & \lambda^2 + (B - j)\lambda + C - k - j(B - j) \\
\hline
\lambda^2 + j\lambda + k & \lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E \\
 & \cdots\cdots\cdots\cdots\cdots \\
\hline
 & \left[C - k - j(B - j)\right]\lambda^2 + \left[D - k(B - j)\right]\lambda + E \\
 & \left[C - k - j(B - j)\right]\lambda^2 + j\left[C - k - j(B - j)\right]\lambda + k\left[C - k - j(B - j)\right] \\
\hline
 & \text{Remainder}
\end{array}$$

If the remainder is equal to zero or negligible, the two quadratics obtained are good approximations to the factors of the quartic equation. If the remainder is not negligible, the procedure is repeated by using as the second approximation the factor $\lambda^2 + j'\lambda + k'$ where

$$j' = \frac{D - k(B - j)}{C - k - j(B - j)}$$

and

$$k' = \frac{E}{C - k - j(B - j)}$$

This operation is continued until the remainder is negligible. The results usually converge rapidly if the frequencies of the two modes are of different orders of magnitude or if the quartic equation has small real roots. The usefulness of the method is greatly reduced in cases where the results converge slowly.

A substitute method has been derived by the authors to be used as a means of obtaining the roots of the quartic equation if the method of reference 5 converges slowly.

It can be seen from the division performed previously that if $\lambda^2 + j\lambda + k$ is a factor of the quartic equation, two parametric equations must be satisfied, namely

$$\left.\begin{aligned} D - k(B - j) &\equiv j\left[C - k - j(B - j)\right] \\ E &\equiv k\left[C - k - j(B - j)\right] \end{aligned}\right\} \tag{B1}$$

The method consists in solving these two equations simultaneously for j and k. A convenient procedure is as follows:

Equations (B1) are rearranged in the form

$$\left.\begin{aligned} k - \frac{j^3 - j^2B + jC - D}{2j - B} &= 0 \\ k^2 - (j^2 - jB + C)k + E &= 0 \end{aligned}\right\} \tag{B2}$$

The first of equations (B2) is readily solved for k for any given values of j. A series of corresponding values of j and k are computed in this way. For each corresponding j and k the value of the left-hand side of the second equation of equations (B2) is evaluated. This value is plotted as ordinate against j as abscissa. The intersection of the curve with the j-axis evidently determines the solution for j. The solution for k is obtained upon substitution of this value of j back into the upper equation. The values of j and k thus obtained may be substituted back into the lower equation to check the solution.

The methods presented are readily adaptable to equations of any even order. In the case where an uneven-order equation should appear, the logical procedure is to obtain a real root by synthetic division and thereby reduce the equation to an even order.

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TABLE I

STABILITY DERIVATIVES AND MASS CHARACTERISTICS

OF HYPOTHETICAL AIRPLANE

W/S, lb/ft^2	40
b, ft	20
ρ, slugs/ft^3	0.0004
V, ft/sec	733
γ, deg	0
C_L	0.372
μ_b	155
k_{X_0}, ft	2.02
k_{Z_0}, ft	9.64
η, deg	2
C_{l_p}, per radian	-0.197
C_{l_r}, per radian	0.0929
C_{n_p}, per radian	-0.00732
C_{n_r}, per radian	$-1.47\ C_{n_{\beta(tail)}}$
C_{Y_p}, per radian	0
C_{Y_r}, per radian	0
C_{Y_β}, per radian	$-1.33\ C_{n_{\beta(tail)}}$
$C_{n_{\beta(fuselage)}}$, per radian	-0.25
C_{n_β}, per radian	Variable
$C_{n_{\beta(tail)}}$, per radian	$C_{n_\beta} - C_{n_{\beta(fuselage)}}$
C_{l_β}, per radian	Variable

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TABLE II

VALUES OF $\dfrac{\epsilon_n}{(V/b)^n}$ AND $\dfrac{\phi_n}{(V/b)^{n-1}}$ WHICH SATISFY THE DAMPING CRITERION

$T_{1/2} = 1.5 \qquad (0 \leqq P \leqq 2)$

$T_{1/2} = 2.5P - 3.5 \qquad (2 \leqq P \leqq \infty)$

P	ϵ_0	$\frac{\epsilon_1}{V/b}$	$\frac{\epsilon_2}{(V/b)^2}$	$\frac{\epsilon_3}{(V/b)^3}$	$\frac{\epsilon_4}{(V/b)^4}$	$\frac{\phi_0}{(V/b)^{-1}}$	ϕ_1	$\frac{\phi_2}{V/b}$	$\frac{\phi_3}{(V/b)^2}$	$\frac{\phi_4}{(V/b)^3}$
1	1	-0.0116	-0.0249	0.00087	0.00061	0	0.997	-0.0232	-0.0246	0.00116
2	1	-.0460	-.0950	.0133	.00821	0	.989	-.0907	-.0898	.01725
3	1	-.0391	-.223	.0263	.0485	0	.997	-.0780	-.220	.0350
4	1	-.0433	-.400	.0521	.157	0	.998	-.0864	-.395	.0690
5	1	-.0486	-.626	.0917	.386	0	.998	-.0969	-.620	.121
6	1	-.0545	-.903	.148	.804	0	.998	-.109	-.895	.197
7	1	-.0625	-1.230	.231	1.493	0	.998	-.125	-1.220	.307
8	1	-.0676	-1.607	.327	2.553	0	.999	-.135	-1.596	.436
9	1	-.0746	-2.035	.457	4.096	0	.999	-.149	-2.021	.607
10	1	-.0814	-2.513	.614	6.248	0	.999	-.162	-2.496	.815

TABLE III

NATURE OF ROOTS OF STABILITY EQUATION

IN C_{n_β}, C_{l_ϕ} PLANE

Point (see fig. 5)	$T_{1/2}$	T_2	P
A	10.65	----	4.95
	3.08	----	1.19
B	9.95	----	7.05
	3.16	----	1.53
C	7.72	----	30.2
	3.50	----	2.16
D	1.02	1.65	Aperiodic
	4.20	----	3.15
E	.793 2.50	----	Aperiodic
	----	2.39	7.93

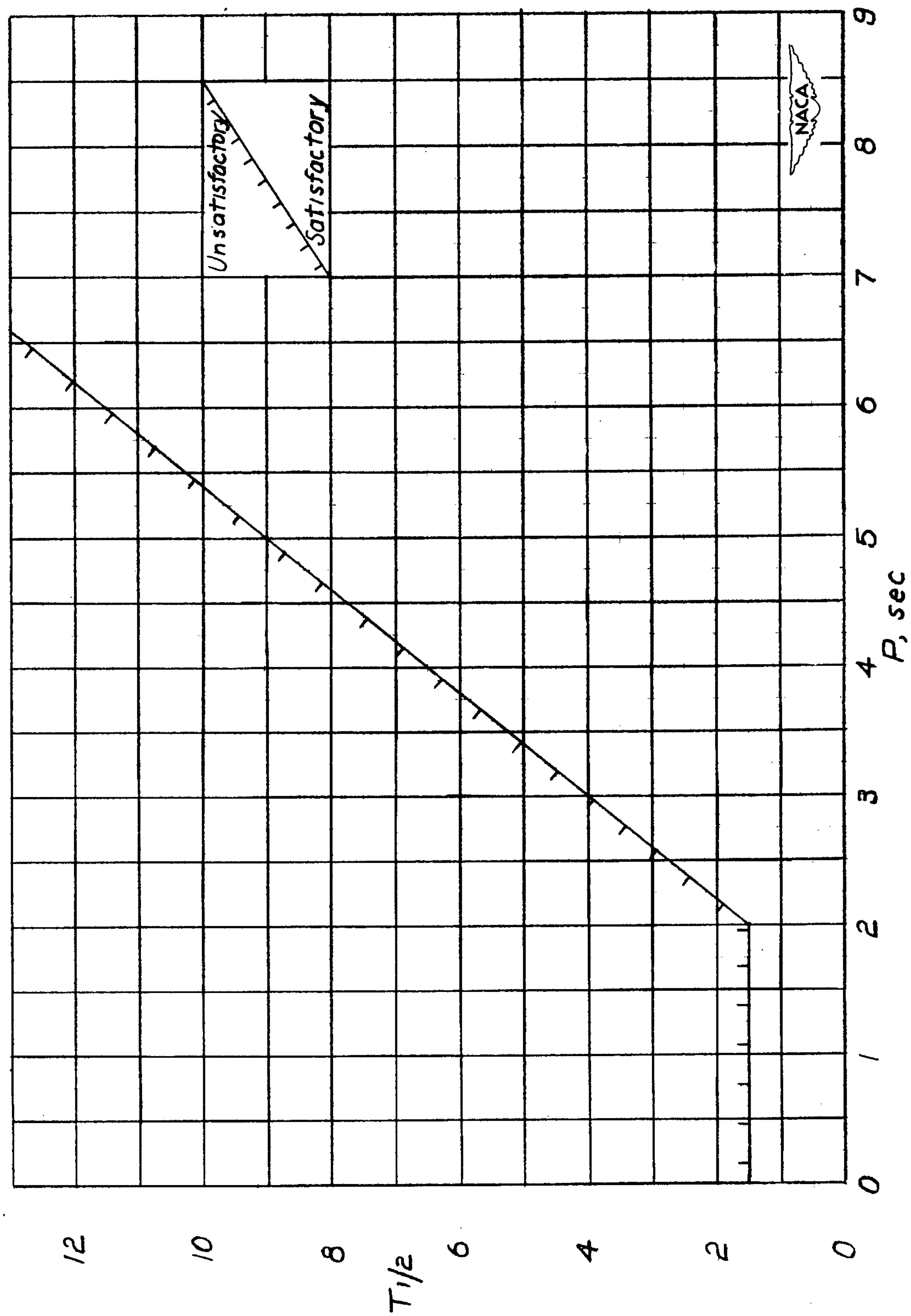

Figure 1.- Navy-Air Force criterion for period-damping relationship of the oscillatory mode of motion.

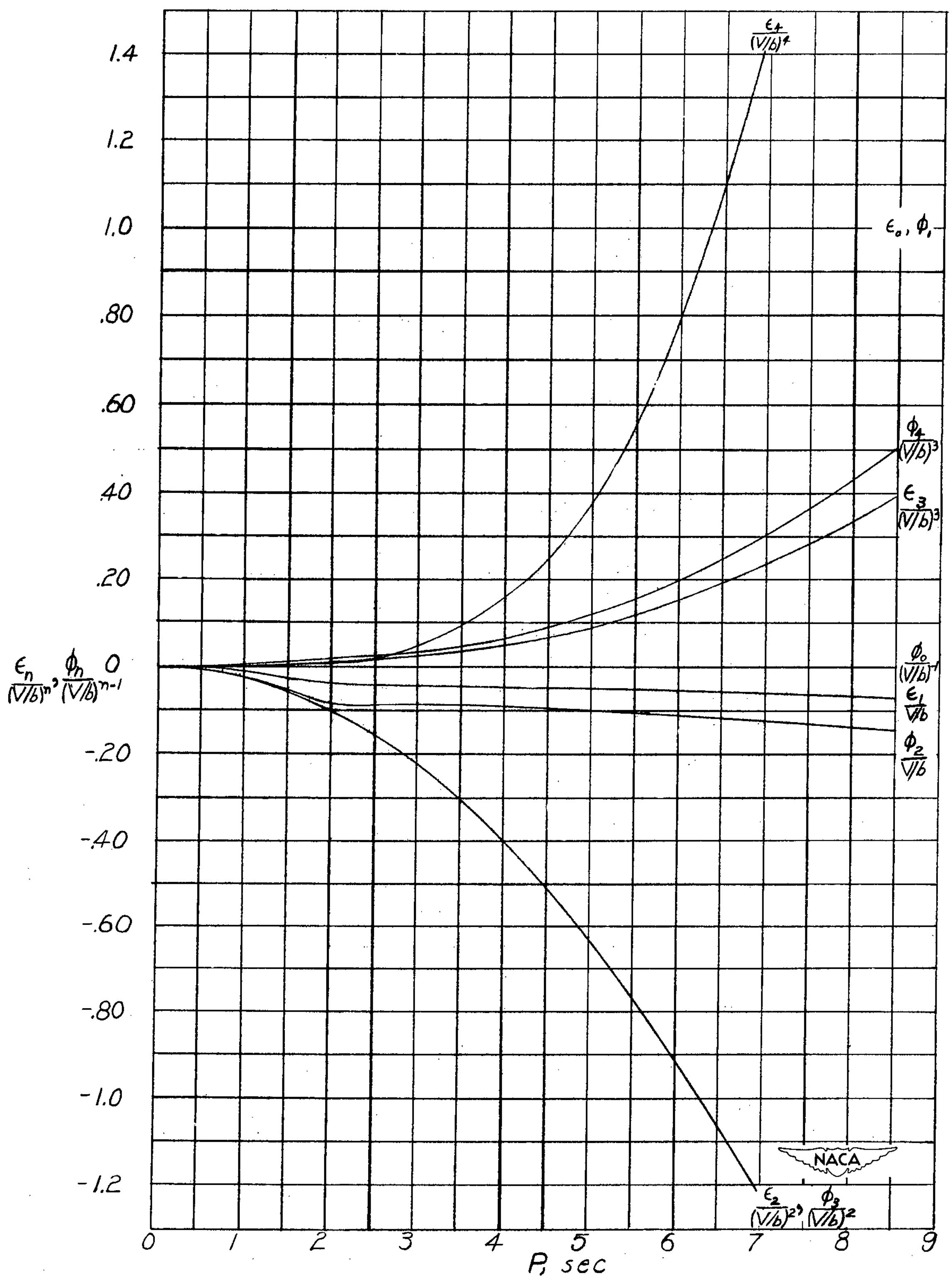

Figure 2.– Variation of the computation factors $\frac{\epsilon_n}{(V/b)^n}$ and $\frac{\phi_n}{(V/b)^{n-1}}$ with P for the Navy-Air Force damping-period criterion. The values of $\frac{\epsilon_n}{(V/b)^n}$ and $\frac{\phi_n}{(V/b)^{n-1}}$ are presented in table II.

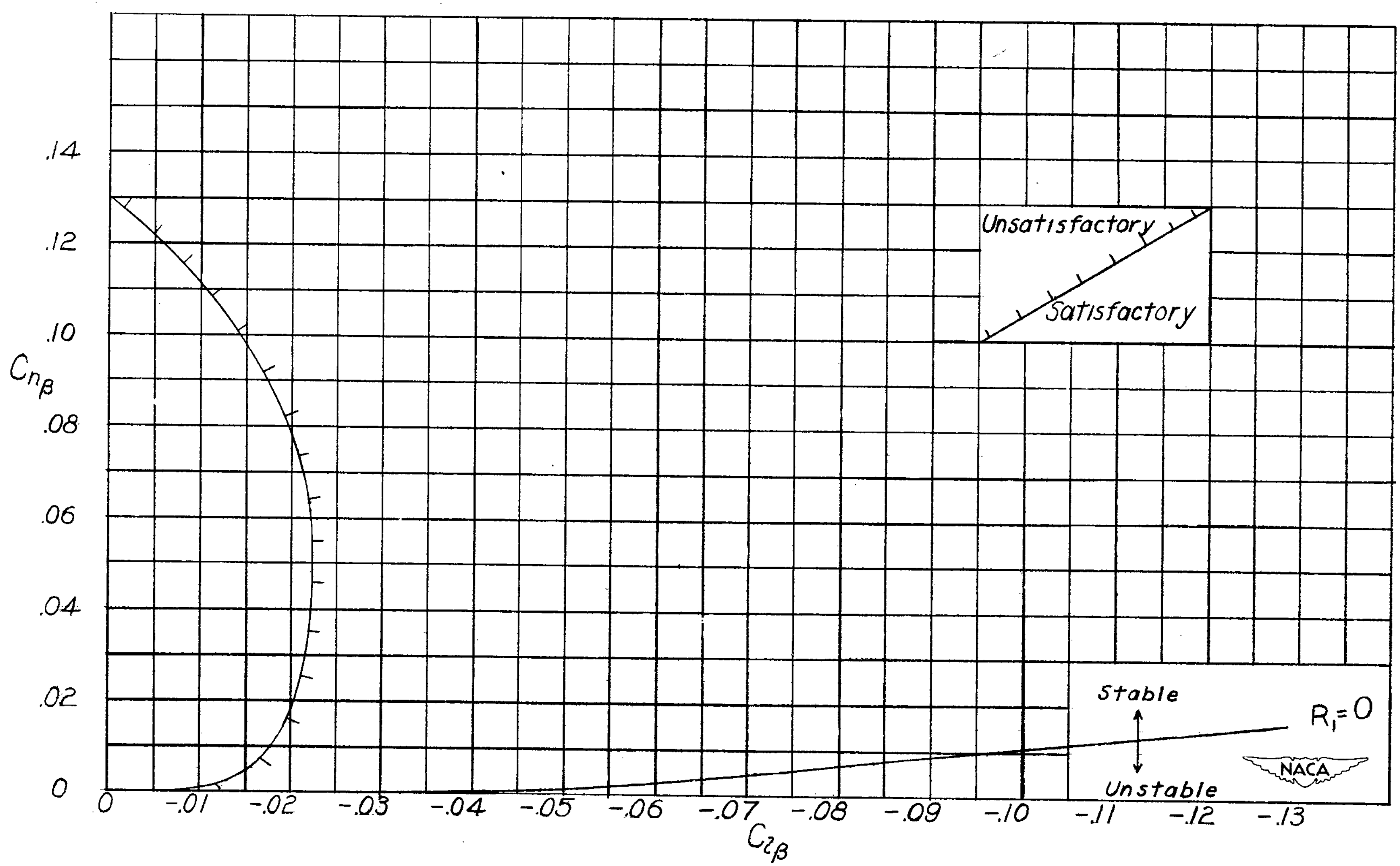

Figure 3.– Boundary for satisfactory relation between damping and period according to Navy–Air Force criterion. $R_1 = 0$ boundary shown for comparison. Hypothetical airplane.

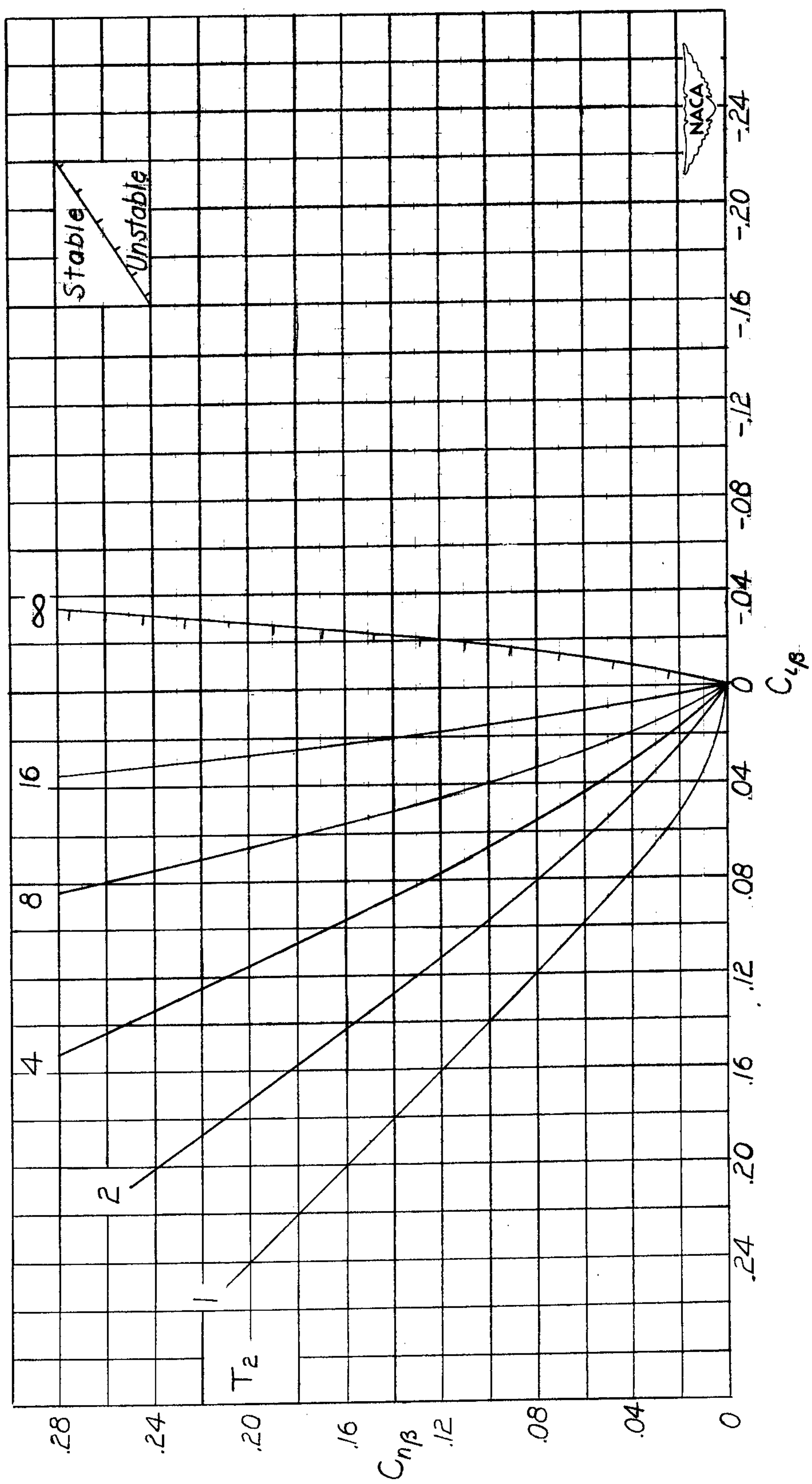

Figure 4.- Curves representing constant rates of spiral divergence. Hypothetical airplane.

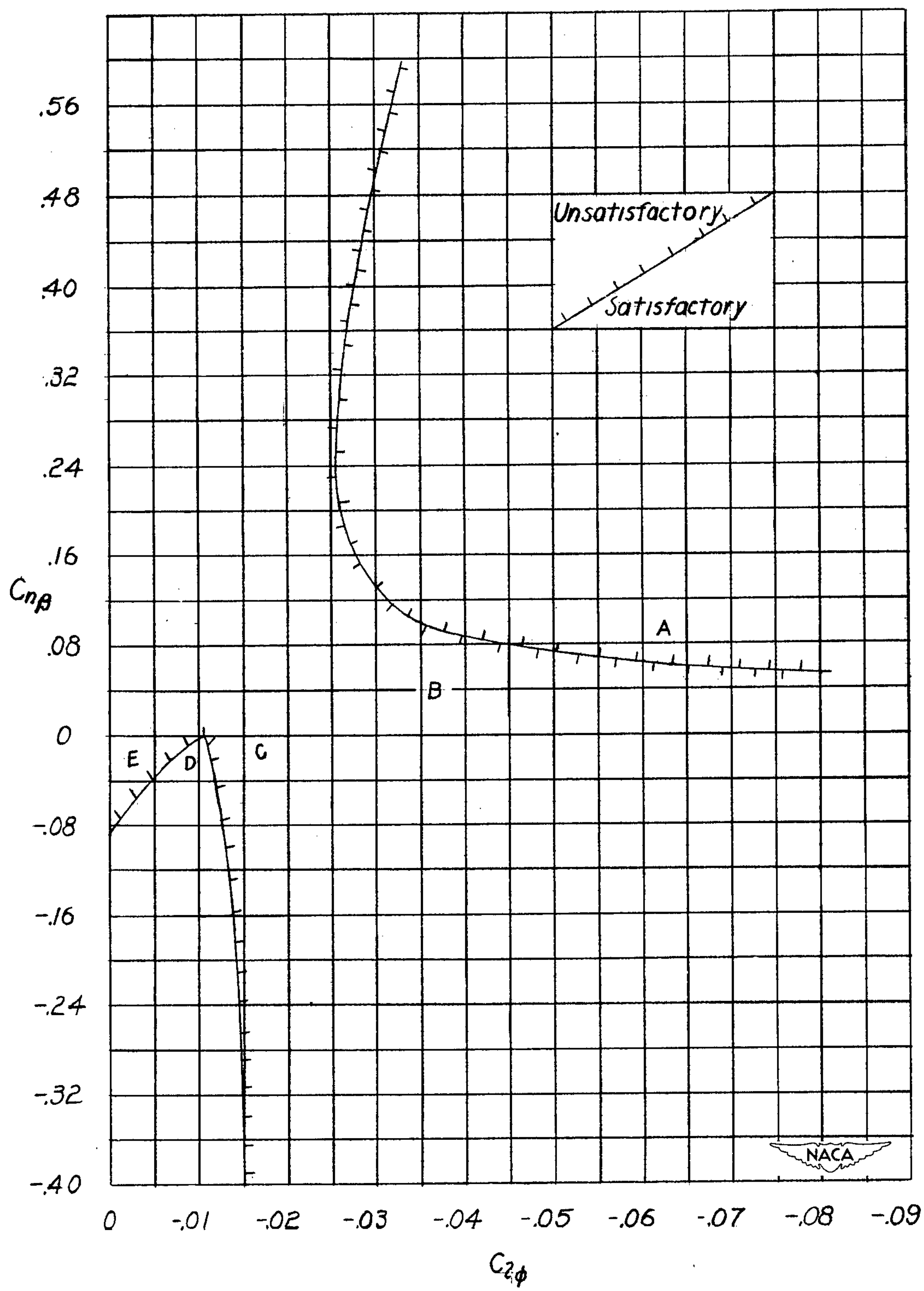

Figure 5.- Boundary defining region of satisfactory period-damping relationship according to Navy-Air Force criterion. Airplane with automatic-pilot coupling of aileron to angle of bank.